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SOME CONVERGENCE RESULTS ON DYNAMIC FACTOR MODELS

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Abstract:

We review some recent papers on a large dynamic factor model (LDFM) and its applications to structural macroeconomic analysis. Then we prove some convergence results concerning with the stochastic variables which define such a model.

Keywords: dynamic factor model, fundamentalness, identification, estimation, consistency, convergence.

JEL Classification: C01, C32, E32.

1. Introduction

Factor models have been used by several authors to address many different economic issue nowadays. Some literature has focused on models specifically designed to handle a large amount of information: the *generalized dynamic factor models* (Forni, Hallin, Lippi and Reichlin, 2000; Forni and Lippi, 2001; Bai and Ng, 2002). Such models have been successfully used for forecasting and, recently, also for structural macroeconomic analysis (Forni, Giannone, Lippi, Reichlin, 2009, FGLR from now on; Forni and Gambetti, 2010).

The main idea underlying factor analysis is that a large set of variables can be explained by a small number of latent variables, the *factors*, which are responsible for all the relevant dynamics. Factor analysis is a technique of dimension reduction that takes the information contained in a large data set and summarizes it by means of few unobservable variables. In this context, it is assumed that macroeconomic observable variables are represented as the sum of two unobservable components, called the *common component* and the *idiosyncratic component*. The common component captures that part of the series which comove with the rest of the economy and the idiosyncratic component is the residual. The idiosyncratic components are not necessarily orthogonal to each other and they are not of direct interest for the analysis since they arise from measurement errors or sectoral sources of variation. The vector of the common components is highly singular, i.e., it is driven by a very small number of shocks (the "common shocks" or "common factors") as compared to the number of variables.

Furthermore, the relation between the common part of the observable series and the factors is assumed to be linear. Structural analysis requires the identification of the macroeconomic shocks and their dynamic effect on macroeconomic variables. The approach is a combination of structural vector autoregression (SVAR) analysis and large-dimensional dynamic factor models. More precisely, the factor model is used to consistenly estimate common and idiosyncratic components of macroeconomic variables. Then the identification of the relationship between common components and macroeconomic shocks can be obtained just in the same way as in SVAR models, and the impulse response functions can be consistently estimated by means of a relatively simple procedure.

In this paper we review some recent papers on a large dynamic factor model and its applications to structural macroeconomic analysis. Then we prove some convergence results concerning with the stochastic variables which define such a model.

2. A Dynamic Factor Model

In this section we illustrate the basic definitions and results concerning with a dynamic factor model, briefly called *the model FGLR*, introduced and studied by Forni, *et al.* (2009).

1) **The model.** Denote by $\mathbf{x}_n^T = (x_{it})_{i=1,\dots,n;\,t=1,\dots,T}$ an $n \times T$ rectangular array of observations, and make two preliminary assumptions:

PA1. *The array $\mathbf{x}_n^T$ is a finite realization of a real-valued stochastic process*

$$X = \{x_{it}: i \in \mathbb{N}, t \in \mathbb{Z}, x_{it} \in L_2(\Omega, \mathcal{F}, P)\}$$

where the n-dimensional vector processes $\{x_{nt} = (x_{1t} \dots x_{nt})' : t \in Z\}_{n \in N}$ are stationary, with zero mean and finite second-order moments $\Gamma_{nk}^x = E(\mathbf{x}_{nt}\mathbf{x}_{n,t-k}')$ for every $k \in N$.

PA2. *For all $n \in N$, the process $\{x_{nt} : t \in Z\}$ admits a Wold representation $\mathbf{x}_{nt} = \sum_{k=0}^{\infty} C_k^n \mathbf{w}_{n,t-k}$, where the full-rank innovations $\mathbf{w}_{nt}$ have finite moments of order four and the matrices $C_k^n = (c_{ij,k}^n)$ satisfy $\sum_{k=0}^{\infty} |c_{ij,k}^n| < \infty$ for all $i, j, n \in N$.*

The model FGLR is obtained by assuming that each variable x_{it} is the sum of two unobservable components $x_{it} = \chi_{it} + \xi_{it}$, where χ_{it} (resp. ξ_{it}) is called the *common* (resp. *idiosyncratic*) component. The common component χ_{it} is driven by q common shocks $\mathbf{u}_t = (u_{1t} \cdots u_{qt})'$ for q independent of n and $q << n$. More precisely:

FM0. *(Dynamic factor structure of the model FGLR)*

Defining $\boldsymbol{\chi}_{nt} = (\chi_{1t} \cdots \chi_{nt})'$ and $\boldsymbol{\xi}_{nt} = (\xi_{1t} \cdots \xi_{nt})'$, suppose that

$$\mathbf{x}_{nt} = \boldsymbol{\chi}_{nt} + \boldsymbol{\xi}_{nt} = B_n(L)\mathbf{u}_t + \boldsymbol{\xi}_{nt} \tag{1.1}$$

where $\mathbf{u}_t$ is a q-dimensional orthonormal white-noise vector, that is, $E(\mathbf{u}_t\mathbf{u}_t') = \mathbf{I}_q$ for all t. The shocks $\mathbf{u}_t$ will be called dynamic factors.

Moreover, assume that

$$B_n(L) = A_n N(L) \tag{1.2}$$

where:

i) $N(L)$ is an $r \times q$ absolutely summable matrix function of L, that is,

$$N(L) = \sum_{k=0}^{\infty} \mathbf{\Psi}_k L^k \qquad \sum_{k=0}^{\infty} |\Psi_{jh,k}| < \infty \qquad \text{for all } j,h$$

where

$$\mathbf{\Psi}_k = (\Psi_{jh,k})_{j=1,\dots r;\, h=1,\dots q}$$

ii) $A_n = (a_{ij})_{i=1,\dots n;\, j=1,\dots r}$ is an $n \times r$ matrix, nested in A_m for all $m > n$.

Defining the $r \times 1$ vector $\mathbf{f}_t = (f_{1t} \cdots f_{rt})'$, called the *static factor*, as

$$\mathbf{f}_t = N(L)\mathbf{u}_t \tag{1.3}$$

Equation (1.1) can be rewritten in the static form

$$\mathbf{x}_{nt} = A_n \mathbf{f}_t + \boldsymbol{\xi}_{nt} \tag{1.4}$$

From (1.1) and (1.4) we get

$$\boldsymbol{\chi}_{nt} = A_n \mathbf{f}_t \tag{1.5}$$

hence

$$\chi_{it} = \sum_{j=1}^{r} a_{ij} f_{jt}.$$

This means that all the variables χ_{it}, $i = 1,\ldots,\infty$, belong to the finite dimensional vector space spanned by $\mathbf{f}_t = (f_{1t} \cdots f_{rt})'$.

Following Forni, *et al.* (2009), we are going to illustrate some conditions under which the shocks $\mathbf{u_t}$ can be identified and estimated by means of the observable variables x_{it}. First, we recall the assumptions necessary for the identification and the estimation of the common components χ_{it}.

FM1. (Orthogonality of common and idiosyncratic components)

For all n*, the vector* $\boldsymbol{\xi}_{nt}$ *is stationary, and* $E(\mathbf{u}_t \boldsymbol{\xi}_{n\tau}') = 0$ *for any* $t, \tau \in Z$ *and* $n \in N$.

Let $\Gamma_{nk}^{x} = E(\mathbf{x}_{nt}\mathbf{x}_{n,t-k}')$, $\Gamma_{nk}^{\chi} = E(\boldsymbol{\chi}_{nt}\boldsymbol{\chi}_{n,t-k}')$ and $\Gamma_{nk}^{\xi} = E(\boldsymbol{\xi}_{nt}\boldsymbol{\xi}_{n,t-k}')$ be the k-lag covariance matrices of $\mathbf{x}_{nt}$, $\boldsymbol{\chi}_{nt}$ and $\boldsymbol{\xi}_{nt}$, respectively. Denote by μ_{nj}^{x}, μ_{nj}^{χ} and μ_{nj}^{ξ} the jth eigenvalues, in decreasing order, of Γ_{n0}^{x}, Γ_{n0}^{χ} and Γ_{n0}^{ξ}, respectively.

FM2. (Pervasiveness of common dynamic and static factors)

a) The complex matrix $N(e^{-i\theta})$ *has (maximum) rank* q *for* θ *almost everywhere in* $[-\pi, \pi]$*;*

b) There exist positive real constants $\underline{c}_j < \bar{c}_j$*,* $j = 1,\ldots,r$*, such that* $\underline{c}_j > \bar{c}_{j+1}$*,* $j = 1,\ldots,r-1$*, and*

$$\underline{c}_j \le \liminf_{n\to\infty} \frac{\mu_{nj}^{\chi}}{n} \le \limsup_{n\to\infty} \frac{\mu_{nj}^{\chi}}{n} \le \bar{c}_j$$

Proposition 1.1 *Under assumption FM2, the* $r \times r$ *matrix* $A_n' A_n$ *has full rank* r *for* n *sufficiently large.*

Assumption FM2 also implies that the common components χ_{it} are identified (see Chamberlain, and Rothschild (1983)), and that the number q is unique, i.e., a representation of type (1.1) - (1.4) with a different number of dynamic factors is not possible (see Forni and Lippi (2001)).

FM3. (Non-pervasiveness of the idiosyncratic components)

There exists a real number d *such that* $\mu_{n1}^{\xi} \le d$ *for any* $n \in N$*. This obviously implies that* $\mu_{nj}^{\xi} \le d$ *for any* $n \in N$ *since such eigenvalues are in decreasing order.*

Assumption FM3, jointly with the identification of the common components χ_{it}, implies that the vector space spanned by the r static factors $f_{1t},\cdots,f_{rt}$ (in $\mathbf{f}_t$) is identified, or, equivalently, $\mathbf{f}_t$ is identified, up to non-singular linear transformation.

In conclusion, given a model of type (1.1)-(1.4), then under assumption FM0-FM3 the integers q and r, the components χ_{it} and ξ_{it}, and the vector space spanned by $\mathbf{f}_t$ are identified.

2) **Fundamentalness**. First we recall briefly some basic notions on fundamental representations of stationary stochastic vectors. Assume that the n stochastic vector $\boldsymbol{\mu}_t$ admits a moving average (MA) representation

$$\boldsymbol{\mu}_t = K(L)\mathbf{v}_t \tag{2.1}$$

where $K(L)$ is an $n \times q$ square-summable filter and $\mathbf{v_t}$ is a q-dimensional white noise.

Definition 2.1 *If $\mathbf{v}_t$ belongs to the vector space spanned by present and past values of $\boldsymbol{\mu}_t$, then the MA in (2.1) is said to be fundamental, and $\mathbf{v}_t$ is called fundamental for $\boldsymbol{\mu}_t$.*

Without loss of generality, we can suppose that $q \leq n$ and that $\mathbf{v}_t$ is full rank. Moreover, for our purpose, we can suppose that the entries of $K(L)$ are rational functions of L and that the rank of $K(z)$, $z \in C$, is maximal, i.e., it is q except for a finite number of complex numbers.

Proposition 2.1 *The MA representation in (2.1) is fundamental if and only if the rank of $K(z)$ is q for all complex numbers z such that $|z|<1$.* For the proof, see Rozanov (1967).

Fundamentalness plays an important role for the identification of the structural shocks in SVAR analysis. SVAR analysis starts with the projection of a full rank q-dimensional vector $\boldsymbol{\mu}_t$ on its past, thus producing an q-dimensional full rank fundamental white-noise $\mathbf{w}_t$. Then the structural shocks are obtained as a linear transformation $A\mathbf{w}_t$, where the matrix A arises from economic theory statements. This is equivalent to assume that the structural shocks are fundamental.

Fundamentalness has here the effect that the identification problem is enormously simplified. However, economic theory, in general, does not provide support for fundamentalness, so that all representations that fulfill the same economic statements, but are not fundamental, are ruled out in SVAR analysis with no justification. Such representations, although they imply the same autocovariance structure, cannot be obtained from inversion of estimated VARs. The situation changes if the structural analysis is conducted assuming that n is large with respect to q. The fundamentalness is also required by dynamic factor models but it is a condition less pressing than in VARs. The first reason is that non fundamentalness of structural shocks arises when the econometrician's information set is smaller than the agent's. The second reason comes from a mathematical background. Precisely, a crucial step in our analysis is the dynamic specification of the common components $\boldsymbol{\chi}_{nt}$ as vector autoregression (VAR) driven by only q macroeconomics shocks $\mathbf{u}_t$, i.e., $\boldsymbol{\chi}_{nt} = A_n N(L)\mathbf{u}_t$, where $q<n$. So the model contains only q variables; suppose such variables are χ_{jt}, $j=1,...,q$, and they cannot ensure fundamentalness of $\mathbf{u}_t$. By Proposition 2.1, the rank of $B_n(z)$ is less than q for some complex number z with $|z|<1$, or equivalently, the polynomials $B_{nj}(z)$, $j=1,...,q$, have a common root. However, the informational advantage of the agents may disappear if the econometrician observes a large set of additional macroeconomic shocks. The generating process of χ_{jt}, $j=q+1,...,n$, contains parameters that do not belong to the generating process of χ_{jt}, $j=1,...,q$, and viceversa. Therefore, with all likelihood, their dynamic responses to $\mathbf{u}_t$ are sufficiently *heterogeneous*, with respect to the first q, to prevent the rank reduction of $B_n(z)$. Now Assumption FM2(b) gives that, for n sufficiently large, A_n has full rank r. Then $N(L)$ has full rank q and it is left-invertible. So the concept of fundamentalness can be adapted to our specification of the dynamic factor model, as follows:

FM4. (*Fundamentalness)*

The matrix function $N(L)$ in (1.2) is left invertible, i.e., there exists an $q \times r$ square-summable filter $G(L)$ such that $G(L)N(L) = \mathbf{I}_q$.

The following proposition shows that FM4, jointly with FM2, imply fundamentalness in the sense of Definition 2.1.

Proposition 2.2 *If FM0-FM4 are satisfied,then $\mathbf{u}_t$ is fundamental for $\boldsymbol{\chi}_{nt}$ for n sufficiently large, and therefore fundamental for $\chi_{it}, i=1,\ldots,\infty$.Moreover, $\mathbf{u}_t$ belongs to the vector space spanned by present and past values of $x_{it}, i=1,\ldots,\infty$, that is, the shocks u_{ht} can be recovered as limits of linear combinations of the variables x_{it}.* For the proof see Forni, *et al.* (2009).

To introduce the last assumption, a VAR specification for $\mathbf{f}_t$, let us consider the orthogonal projection of $\mathbf{f}_t$ on the space spanned by its past values

$$\mathbf{f}_t = Proj(\mathbf{f}_t \mid \mathbf{f}_{t-1}, \mathbf{f}_{t-2}, \ldots) + \mathbf{w}_t \tag{2.2}$$

where $\mathbf{w}_t$ is the r-dimensional vector of the residuals. Under our assumptions, $\mathbf{w}_t$ has rank q. Moreover, assumption FM4 implies that $\mathbf{w}_t = R\mathbf{u}_t$, where R is a maximum-rank $r \times q$ matrix. In the sequel we will adopt the VAR(p) specification:

FM4'. *(Fundamentalness: VAR(p) specification)*

The r - dimensional static factors $\mathbf{f}_t$ admit a VAR(p) representation

$$\mathbf{f}_t = D_1\mathbf{f}_{t-1} + \cdots + D_p\mathbf{f}_{t-p} + R\mathbf{u}_t \tag{2.3}$$

where D_i is $r \times r$ and R is a maximum-rank matrix of dimension $r \times q$.

By (2.3) we have

$$\mathbf{f}_t = (I - D_1L - \cdots - D_pL^p)^{-1}R\mathbf{u}_t$$

Hence

$$\chi_{nt} = A_n(I - D_1L - \cdots - D_pL^p)^{-1}R\mathbf{u}_t$$

by (1.5). So Equation (1.1) yields

$$B_n(L) = A_n(I - D_1L - \cdots - D_pL^p)^{-1}R \tag{2.4}$$

called the *impulse-response function* (IRF) of the lags L.

3) Estimation. The following procedure of estimation can be found in Forni and Gambetti (2010) (see also Forni, *et al.* (2009) Sect.4.2).

Step 1) First, we need to set value for the number r of the static factors $\mathbf{f}_t = (f_{1t} \cdots f_{rt})'$. Bai and Ng (2002) proposed some consistent criteria to determine an estimation of r. Let $\hat{r}$ denote an estimation of r obtained by such criteria.

Step 2) We estimate the static factors $\mathbf{f}_t$, up to a non-singular linear transformation, by means of the first $\hat{r}$ ordinary principal components of the variables $\mathbf{x}_{nt}$ in the data set. Setting

$$\hat{\Gamma}_k^x = \frac{1}{T}\sum_{h=k+1}^{T} \mathbf{x}_{nt}\mathbf{x}_{n,t-k}'$$

and $\hat{\mu}_j^x$ the jth greatest eigenvalue of the sample variance matrix $\hat{\Gamma}_0^x$, the ordinary principal components method gives

$$\mathbf{f}_t = (\hat{f}_{1t} \cdots \hat{f}_{\hat{r}t})' = \hat{A}_n'\mathbf{x}_{nt} = \begin{pmatrix} \hat{a}_{11} & \cdots & \hat{a}_{n1} \\ \hat{a}_{12} & \cdots & \hat{a}_{n2} \\ \vdots & & \vdots \\ \hat{a}_{1\hat{r}} & \cdots & \hat{a}_{n\hat{r}} \end{pmatrix} \begin{pmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{pmatrix}$$

where $\hat{A}_n$ is the $n \times \hat{r}$ matrix having on the jth column the normalized eigenvector $\hat{a}_j = (\hat{a}_{1j}\hat{a}_{2j}\cdots\hat{a}_{nj})'$ corresponding to $\hat{\mu}_j^x$ for $j = 1,\ldots,\hat{r} \le n$, hence

$$\hat{A}_n = \begin{pmatrix} \hat{a}_{11} & \cdots & \hat{a}_{1\hat{r}} \\ \hat{a}_{21} & \cdots & \hat{a}_{2\hat{r}} \\ \vdots & & \vdots \\ \hat{a}_{n1} & \cdots & \hat{a}_{n\hat{r}} \end{pmatrix} = (\hat{a}_1'\hat{a}_2'\cdots\hat{a}_{\hat{r}}')'.$$

Step 3) We set a number of lags $\hat{p}$ and run a VAR($\hat{p}$) as in (2.3) with the estimated static factors $\hat{\mathbf{f}}_t$ to get estimates $\hat{D}(L)$ and $\hat{\boldsymbol{\varepsilon}}_t$ of the matrix function $D(L)$ and the residuals $\boldsymbol{\varepsilon}_t$, respectively. Recall that $\varepsilon_t = R\mathbf{u}_t$ in (2.3).

Step 4) We estimate the number q of the dynamic factors $\mathbf{u}_t = (u_{1t}\cdots u_{qt})'$ obtained by using three criteria which were described in Bai and Ng (2007), Stock and Watson (2005), and Onastki (2009), respectively. Denote this estimate by $\hat{q}$.

Step 5) Now let $\hat{\Gamma}^{\varepsilon}$ denote the sample variance-covariance matrix of the estimated residuals $\hat{\boldsymbol{\varepsilon}}_t$. Having an estimate $\hat{q}$ of the number of dynamic factors $\mathbf{u}_t = (u_{1t}\cdots u_{\hat{q}t})'$, we obtain an estimate of a non-structural representation of the common components by using the spectral decomposition $\hat{\Gamma}^{\varepsilon}$. More precisely, let $\hat{\mu}_j^{\varepsilon}$, $j = 1,\ldots,\hat{q}$ be the jth eigenvalue of $\hat{\Gamma}^{\varepsilon}$, taken in decreasing order, $|\hat{\mu}_1^{\varepsilon}| > \ldots > |\hat{\mu}_{\hat{q}}^{\varepsilon}|$. Let $\hat{M} = Diag(\sqrt{\hat{\mu}_j^{\varepsilon}})$ be the $\hat{q} \times \hat{q}$ diagonal matrix with $\sqrt{\hat{\mu}_j^{\varepsilon}}$ as its (j, j)-entry, and $\hat{K}$ the $\hat{r} \times \hat{q}$ matrix having on the columns the normalized eigenvectors corresponding to $\hat{\mu}_1^{\varepsilon},\ldots,\hat{\mu}_{\hat{q}}^{\varepsilon}$. Then the spectral decomposition states $\hat{\Gamma}^{\varepsilon} = \hat{K}\hat{M}\hat{M}'\hat{K}' = \hat{S}\hat{S}'$, where $\hat{S} = \hat{K}\hat{M}$. Thus our estimated matrix of non-structural impulse-response functions in (2.4) is $\hat{C}_n(L) = \hat{A}_n(\hat{D}(L))^{-1}\hat{S}$. Recall that by definition we have $C_n(L) = B_n(L)H$ and $B_n(L) = A_n D(L)^{-1}R$, where $R = SH'$.

Step 6) Finally, we obtain $\hat{H}$ by imposing our identification restrictions on $\hat{B}_m(L) = \hat{C}_m(L)\hat{H}'$. Thus we get estimates

$$\hat{R} = \hat{S}\hat{H}' \qquad\qquad \hat{B}_n(L) = \hat{A}_n\hat{D}(L)^{-1}\hat{R}. \tag{3.1}$$

3. Consistency

Consistency of $\hat{B}_n(L) = \hat{A}_n\hat{D}(L)^{-1}\hat{R}$ as estimator of the impulse-response function $B_n(L)$ for large cross-sections and large sample size, that is, $n, T \to \infty$, was proved in Forni et al.(2009), Proposition 3. For this, it is necessary to state a last assumption.

FM5. *Denote by* $\gamma_{ij,k}^x$ *and* $\hat{\gamma}_{ij,k}^x$ *the* (i, j)*-entries of* Γ_k^x *and* $\hat{\Gamma}_k^x$, *respectively. There exists a positive real number* ρ *such that*

$$T\,E[(\hat{\gamma}_{ij,k}^x - \gamma_{ij,k}^x)^2] < \rho$$

for $k = 0,1$ *and for all positive integers* i, j *and* T.

Proposition 3.1 *Let $\hat{b}_{ni}(L)$ and $b_{ni}(L)$ denote the i - th rows of the matrix functions $\hat{B}_n(L)$ and $B_n(L)$, respectively. Under assumptions PA1-2 and FM1-5, $\hat{b}_{ni}(L)$, for a fixed i, is a consistent estimator of $b_{ni}(L)$, that is,*

$$\underset{\delta_{nt}\to\infty}{plim}\,\hat{b}_{ni}(L) = b_{ni}(L)$$

where $\delta_{nt} = min(n,T)$, n is the number of variables, and T is the number of observations over time.

For the proof see Forni, *et al.*(2009).

Proposition 3.1 states that consistency is achieved along any path for (n,T) with n and T both tending to infinity. The consistency rate is given by $\sqrt{\delta_{nt}} = min(\sqrt{n},\sqrt{T})$. This implies that if the cross-section dimension n is large relative to the sample size T, that is, $T/n \to 0$, the rate of consistency is $\sqrt{T}$, the same we would obtain if the common components were observed, that is, if the variables were not contamined by idiosyncratic components. On the other hand, if $n/T \to 0$, then the consistency rate is $\sqrt{n}$ reflecting the fact that the common components are not observed but have to be estimated.

Here we give a simplified proof of Proposition 3.1. For an $m\times n$ real matrix $\mathbf{A} = (a_{ij})$, the matrix norm $\|\mathbf{A}\|$ of $\mathbf{A}$ is defined as $\|\mathbf{A}\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{tr(\mathbf{A}'\mathbf{A})}$. Let $\mathbf{E}$ and $\mathbf{F}$ be two $n\times n$ symmetric matrices and denote by $\sigma_j(\cdot)$, $j = 1,\ldots,n$, the eigenvalues in decreasing order of magnitude. We shall use the well--known inequalities due to Weyl:

$$|\sigma_j(\mathbf{E}+\mathbf{F}) - \sigma_j(\mathbf{E})| \le \sqrt{\sigma_1(\mathbf{F}^2)} \le \sqrt{tr(\mathbf{F}^2)} = \|\mathbf{F}\|.$$

Denote by Λ_n and $\hat{\Lambda}_n$, the $r\times r$ diagonal matrices having on the diagonal elements the first r largest eigenvalues of $\Gamma_{n0}^{\chi} = E(\boldsymbol{\chi}_{nt}\boldsymbol{\chi}_{nt}')$ and $\hat{\Gamma}_{n0}^{x} = E(\hat{\mathbf{x}}_{nt}\hat{\mathbf{x}}_{nt}')$, respectively. Let us recall here our notation for the eigenvalues of the relevant matrices:

$$\mu_{nj}^{x} := \sigma_j(\Gamma_{n0}^{x}), \hat{\mu}_{nj}^{x} := \sigma_j(\hat{\Gamma}_{n0}^{x}), \mu_{nj}^{\chi} := \sigma_j(\Gamma_{n0}^{\chi}) \quad \text{and} \quad \mu_{nj}^{\xi} := \sigma_j(\Gamma_{n0}^{\xi}).$$

Hence we have $\Lambda_n = Diag\,(\mu_{n1}^{\chi},\cdots,\mu_{nr}^{\chi})$ and $\hat{\Lambda}_n = Diag(\hat{\mu}_{n1}^{x},\cdots,\hat{\mu}_{nr}^{x})$.

From (1.5) and Step 2, Section 2, A_n and $\hat{A}_n$ are the $n\times r$ matrices having on the columns the normalized eigenvectors corresponding to μ_{nj}^{χ} and $\hat{\mu}_{nj}^{x}$ for $j = 1,\cdots,r \le n$, hence we have $\Gamma_{n0}^{\chi} = A_n\Lambda_n A_n'$ and $\hat{\Gamma}_{n0}^{x} = \hat{A}_n\hat{\Lambda}_n\hat{A}_n'$.

Lemma 3.2 *Under assumptions PA1-2 and FM0-5, as $n,T\to\infty$, we have*

(i) $\|\hat{\Gamma}_{n0}^{x} - \Gamma_{n0}^{x}\|^2 = tr[(\hat{\Gamma}_{n0}^{x} - \Gamma_{n0}^{x})^2] = O_p(\frac{n^2}{T})$

(ii) $\frac{1}{n}\hat{\mu}_{nj}^{x} = \frac{1}{n}\mu_{nj}^{\chi} + O_p(\frac{1}{n}) + O_p(\frac{1}{\sqrt{T}})$ for $j = 1,\cdots,r$

Proof. By assumption FM5 there exists a positive constant ρ such that for all $T \in N$ and $i.j \in N$, $T\,E[(\hat{\gamma}^x_{ij,0} - \gamma^x_{ij,0})^2] < \rho$ as $T \to \infty$, where $\hat{\gamma}^x_{ij,0}$ and $\gamma^x_{ij,0}$ denote the (i, j)-entries of $\hat{\Gamma}^x_{n0}$ and Γ^x_{n0}, respectively. We have

$$tr[(\hat{\Gamma}^x_{n0} - \Gamma^x_{n0})^2] = \sum_{i=1}^{n}\sum_{j=1}^{n}(\hat{\gamma}^x_{ij,0} - \gamma^x_{ij,0})^2.$$

Taking expectations yields

$$0 \le E\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\hat{\gamma}^x_{ij,0} - \gamma^x_{ij,0})^2\right] = \sum_{i=1}^{n}\sum_{j=1}^{n}E[(\hat{\gamma}^x_{ij,0} - \gamma^x_{ij,0})^2] < \frac{\rho n^2}{T}$$

hence

$$\| \hat{\Gamma}^x_{n0} - \Gamma^x_{n0} \|^2 = O_p\left(\frac{n^2}{T}\right) \quad \text{which proves (i).}$$

Turning to (ii), from the Weyl inequality, we have (use also(i)):

$$(\hat{\mu}^x_{nj} - \mu^x_{nj})^2 \le tr[(\hat{\Gamma}^x_{n0} - \Gamma^x_{n0})^2] = O_p\left(\frac{n^2}{T}\right)$$

hence

$$0 \le | \hat{\mu}^x_{nj} - \mu^x_{nj} | \le \sqrt{tr[(\hat{\Gamma}^x_{n0} - \Gamma^x_{n0})^2]} = O_p\left(\frac{n}{\sqrt{T}}\right).$$

Moreover, from assumptions FM0-3

$$\frac{1}{n}\mu^x_{nj} = \frac{1}{n}\mu^\chi_{nj} + \frac{1}{n}\mu^\xi_{nj} \le \frac{1}{n}\mu^\chi_{nj} + \frac{1}{n}\mu^\xi_{n1} \le \frac{1}{n}\mu^\chi_{nj} + \frac{d}{n} = \frac{1}{n}\mu^\chi_{nj} + O_p\left(\frac{1}{n}\right)$$

since $\mu^\xi_{nj} \le \mu^\xi_{n1} \le d$ by FM3. Then we have:

$$\frac{1}{n}(\hat{\mu}^x_{nj} - \mu^\chi_{nj}) = \frac{1}{n}(\hat{\mu}^x_{nj} - \mu^x_{nj}) + \frac{1}{n}(\mu^x_{nj} - \mu^\chi_{nj}) = O_p\left(\frac{n}{\sqrt{T}}\right) + O_p\left(\frac{1}{n}\right) \quad \text{which proves (ii).}$$

From Section 2, Step 6, we have:

$$\hat{B}_n(L) = \hat{A}_n\hat{D}(L)^{-1}\hat{R}_n = \hat{A}_n(I + \hat{D}_nL + (\hat{D}_n)^2L^2 + \cdots)\hat{R}_n$$

and

$$B_n(L) = A_n(I + D_nL + D_n{}^2L^2 + \cdots)R_n.$$

Here we assume for simplicity the VAR specification with one lag, the extension to a finite number of lags being immediate.

Lemma 3.3 *Under assumptions PA1-2 and FM0-5, as* $n, T \to \infty$*, we have:*

(i) $\frac{1}{n}\| \hat{\Lambda}_n - \Lambda_n \| = O_p(\frac{1}{n}) + O_p(\frac{1}{\sqrt{T}})$

(ii) $\| A_n^{'}\hat{A}_n - I_r \| = O_p(\frac{1}{\sqrt{n}}) + O_p(\frac{1}{\sqrt{T}})$

(iii) $\| \hat{D}_n - D_n \| = O_p(\frac{1}{\sqrt{n}}) + O_p(\frac{1}{\sqrt{T}})$

(iv) $\| \hat{R}_n - R_n \| = O_p(\frac{1}{\sqrt{n}}) + O_p(\frac{1}{\sqrt{T}})$

Proof. By Lemma 3.2 (ii) we have:

$$\frac{1}{n}\| \hat{\Lambda}_n - \Lambda_n \| = \frac{1}{n}\sqrt{\sum_{j=1}^{r}(\hat{\mu}_{nj}^{x} - \mu_{nj}^{\chi})^2} = O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{T}}\right).$$

Turning to (ii) we have the following decomposition:

$$\begin{aligned}\frac{1}{n}\hat{\Lambda}_n &= \frac{1}{n}\hat{A}_n^{'}\hat{\Gamma}_{n0}^{x}\hat{A}_n = \frac{1}{n}\hat{A}_n^{'}(\Gamma_{n0}^{x} + \hat{\Gamma}_{n0}^{x} - \Gamma_{n0}^{x})\hat{A}_n \\ &= \frac{1}{n}\hat{A}_n^{'}\Gamma_{n0}^{x}\hat{A}_n + \frac{1}{n}\hat{A}_n^{'}(\hat{\Gamma}_{n0}^{x} - \Gamma_{n0}^{x})\hat{A}_n \\ &= \frac{1}{n}\hat{A}_n^{'}(\Gamma_{n0}^{\chi} + \Gamma_{n0}^{\xi})\hat{A}_n + \frac{1}{n}\hat{A}_n^{'}(\hat{\Gamma}_{n0}^{x} - \Gamma_{n0}^{x})\hat{A}_n \\ &= \frac{1}{n}\hat{A}_n^{'}A_n\Lambda_n A_n^{'}\hat{A}_n + \frac{1}{n}\hat{A}_n^{'}\Gamma_{n0}^{\xi}\hat{A}_n + \frac{1}{n}\hat{A}_n^{'}(\hat{\Gamma}_{n0}^{x} - \Gamma_{n0}^{x})\hat{A}_n.\end{aligned}$$

From Lemma 3.2 (i) we get:

$$\frac{1}{n}\| \hat{A}_n^{'}(\hat{\Gamma}_{n0}^{x} - \Gamma_{n0}^{x})\hat{A}_n \| \le \frac{1}{n}\sqrt{tr[(\hat{\Gamma}_{n0}^{x} - \Gamma_{n0}^{x})^2]} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Moreover, we have:

$$\begin{aligned}\frac{1}{n}\| \hat{A}_n^{'}\Gamma_{n0}^{\xi}\hat{A}_n \| &\le \frac{1}{n}\sqrt{tr(\Gamma_{n0}^{\xi})^2} = \frac{1}{n}\sqrt{\sum_{j=1}^{n}(\mu_{nj}^{\xi})^2} \\ &\le \frac{1}{n}\sqrt{n}|\mu_{n1}^{\xi}| \le \frac{d}{\sqrt{n}} = O\left(\frac{1}{\sqrt{n}}\right)\end{aligned}$$

by assumption FM3. The statement (ii) follows. The statements (iii) and (iv) can be proved by similar arguments.

Proof of Proposition 3.1. By Lemma 3.3 (ii) $A_n^{'}\hat{A}_n$ converges to I_r as $n,T \to \infty$ hence $\hat{A}_n$ converges to A_n as $n,T \to \infty$. By Lemma 3.3 (iv) $\hat{R}_n$ converges to R_n as $n,T \to \infty$. Continuity of the matrix product (notice that D_n has fixed dimension r), implies:

$$(\hat{D}_n)^h = (D_n)^h + O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right)$$

hence $(\hat{D}_n)^h$ converges to $(D_n)^h$, as $n,T \to \infty$, for any $h \in N$.

Thus the matrix function $\hat{B}_n(L)$ converges entry-by-entry to $B_n(L)$ as $n,T \to \infty$. This completes the proof of the consistency.

4. Further Convergence Results

The following result states that $\hat{\chi}_{nt}$ converges to χ_{nt} in mean square. See Forni et al.(2000), Proposition (2), for a different proof.

Proposition 4.1

$$\underset{\delta_{nt}\to\infty}{plim}\ var(\chi_{nt}-\hat{\chi}_{nt})=0$$

Proof. Recall that $\chi_{nt}=B_n(L)\mathbf{u}_t$, where $B_n(L)=A_nN(L)$ and $N(L)$ (and hence $B_n(L)$) is an $r\times q$ (resp. $n\times q$) absolutely summable matrix function of L by FM0. So we can set $\chi_{nt}=\sum_{k=0}^{\infty}b_{nk}\mathbf{u}_{t-k}$, where $b_{nk}(L)$ is an $n\times q$ absolutely summable matrix function. Further, $\mathbf{u}_t=(u_{1t}\cdots u_{qt})'$ is an orthonormal vector white noise, hence $E(\mathbf{u}_t\mathbf{u}_t')=I_q$ and $E(\mathbf{u}_t\mathbf{u}_\tau')=0$ for $t\neq\tau$. Then we have:

$$\begin{aligned}var(\chi_{nt}-\hat{\chi}_{nt}) &= E(\chi_{nt}-\hat{\chi}_{nt})(\chi_{nt}-\hat{\chi}_{nt})^{'}\\ &= E(\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}(b_{nk}-\hat{b}_{nk})\mathbf{u}_{t-h}\mathbf{u}_{t-k}^{'}(b_{nk}-\hat{b}_{nk})^{'})\\ &= \sum_{h=0}^{\infty}(b_{nk}-\hat{b}_{nk})(b_{nk}-\hat{b}_{nk})^{'}.\end{aligned}$$

The expectation operator can be moved inside summation because the considered matrix series are absolutely summable. Now the result follows from Proposition 3.1.

This implies that $\hat{\chi}_{nt}$ is a consistent estimator of χ_{nt}, that is, $\underset{\delta_{nt}\to\infty}{plim}\hat{\chi}_{nt}=\chi_{nt}$. The fact $\hat{\chi}_{nt}-\chi_{nt}\xrightarrow{p}0$ gives also the following result on the convergence in distribution: if $\hat{\chi}_{nt}\xrightarrow{L}N(\mu,\sigma^2\mathbf{I})$, then $\chi_{nt}\xrightarrow{L}N(\mu,\sigma^2\mathbf{I})$.

Corollary 4.2

$$\underset{\delta_{nt}\to\infty}{plim}\ var(\mathbf{x}_{nt}-\hat{\chi}_{nt})=E(\xi_{nt}\xi_{nt}^{'})=\Gamma_{n0}^{\xi}$$

Proof. We have:

$$\begin{aligned}&var(x_{nt}-\hat{\chi}_{nt})=E(\chi_{nt}+\xi_{nt}-\hat{\chi}_{nt})(\chi_{nt}^{'}+\xi_{nt}^{'}-\hat{\chi}_{nt}^{'})\\ &=var(\chi_{nt}-\hat{\chi}_{nt})+E((\chi_{nt}-\hat{\chi}_{nt})\xi_{nt}^{'})+E(\xi_{nt}(\chi_{nt}^{'}-\hat{\chi}_{nt}^{'}))+E(\xi_{nt}\xi_{nt}^{'}).\end{aligned}$$

Now the idiosyncratic components ξ_{it} are orthogonal to the $\mathbf{u}$'s at any lead and lag. Thus the second and the third summands vanish as n, T go to infinity. By Proposition 4.1, we get the result.

The matrix function $N(L)$, where $B_n(L)=A_nN(L)$, is left invertible (fundamentalness), that is, there exists an $q\times r$ square-summable filter $G(L)$ such that $G(L)N(L)=I_q$. Then we have:

Proposition 4.3 *An estimator $\hat{\chi}_{nt}$ of the common components χ_{nt} can be obtained asymptotically from the sequence $K_n(L)\mathbf{x}_{nt}$, where the square-summable filter $K_n(L)$ is given by $K_n(L)=B_n(L)G(L)\hat{A}_n^{'}$. More precisely, we have:*

$$\underset{\delta_{nt}\to\infty}{plim}\ K_n(L)\mathbf{x}_{nt}=\hat{\chi}_{nt}$$

Proof. Multiplying by $\hat{A}_n^{'}$ on the left the equation:

$$\mathbf{x}_{nt} = \boldsymbol{\chi}_{nt} + \boldsymbol{\xi}_{nt} = B_n(L)\mathbf{u}_t + \boldsymbol{\xi}_{nt} = A_n N(L)\mathbf{u}_t + \boldsymbol{\xi}_{nt}$$

Yields

$$\hat{A}_n^{'}\mathbf{x}_{nt} = \hat{A}_n^{'} A_n N(L)\mathbf{u}_t + \hat{A}_n^{'}\boldsymbol{\xi}_{nt}.$$

By Lemma 3.3(ii) $\hat{A}_n^{'} A_n \xrightarrow{p} I_r$ as n,T go to infinity. Hence for n,T sufficiently large, we can write:

$$\hat{A}_n^{'}\mathbf{x}_{nt} = N(L)\mathbf{u}_t + \hat{A}_n^{'}\boldsymbol{\xi}_{nt}.$$

Thus, for n,T sufficiently large, we get:

$$G(L)\hat{A}_n^{'}\mathbf{x}_{nt} = G(L)N(L)\mathbf{u}_t + G(L)\hat{A}_n^{'}\boldsymbol{\xi}_{nt}$$

hence

$$\mathbf{u}_t = G(L)\hat{A}_n^{'}\mathbf{x}_{nt} - G(L)\hat{A}_n^{'}\boldsymbol{\xi}_{nt}$$

and

$$\boldsymbol{\chi}_{nt} = B_n(L)\mathbf{u}_t = B_n(L)G(L)\hat{A}_n^{'}\mathbf{x}_{nt} - B_n(L)G(L)\hat{A}_n^{'}\boldsymbol{\xi}_{nt}.$$

So, for n,T sufficiently large, we obtain:

$$\hat{\boldsymbol{\chi}}_{nt} = B_n(L)G(L)\hat{A}_n^{'}\mathbf{x}_{nt} = K_n(L)\mathbf{x}_{nt}$$

where $K_n(L) = B_n(L)G(L)\hat{A}_n^{'}$.

From above, we see that the filters $K_{ni}(L)$ can be obtained by their empirical counterparts based on finite realizations of the form $\mathbf{x}_n^T = (x_{n1}\cdots x_{nT})^{'}$. Thus we can write:

$$\hat{K}_n(L) = \hat{B}_n(L)\hat{G}(L)\hat{A}_n^{'}.$$

From the above consistency results, it follows:

$$\underset{\delta_{nt}\to\infty}{plim}\ \hat{K}_n(L) = K_n(L).$$

Finally, the process $\left\{\hat{\chi}_{nt} : t \in Z\right\}$ admits a Wold representation for n and T sufficiently large. Since $\mathbf{x}_{nt}$ has a Wold representation as in (PA2), the process $\left\{\hat{\chi}_{nt} : t \in Z\right\}$ asymptotically can be expressed as

$$K_n(L)\mathbf{x}_{nt} = \sum_{k=0}^{\infty} K_{nk}\mathbf{x}_{n,t-k} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} K_{nk} C_h^n \mathbf{w}_{n,t-k-h}.$$

5. Conclusion

In this paper we have proved some convergence results concerning with stochastic variables which define our dynamic factor model. Our convergence results show the appropriate statistical properties that qualify such a model. The factor model enables us to handle a large amount of information and then it avoids important limitations of structural VAR models. For this reason, it is an important tool to be used for economic and financia applications.

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